ON INDEPENDENT SET POLYNOMIALS OF GRAPHS

By

Walid A. M. Saeed
Department of Mathematics
College of Science
University of Taiz ,Yemen

ABSTRACT

The concept of independent vertex palindromic graphs was put forward in a recent paper [1][2].In this paper we will describe certain results of independent vertex unimodal or palindromic graphs with respect to their independence polynomial.

1. Introduction:

In this paper , we consider finite non-trivial graphs without loop or multiple edges . For undefined concepts and notations, see [1],[2],[3].

Let $G = (V, E, i)$ be a graph with number of vertices $|V| = p$ and number of edges $|E| = q$ and
$i : V\times V \rightarrow \{0,1\}$; $u,v\in V$ are independent pair if $i(u,v) = 0$; that is, the two distinct vertices $u,v\in V$ are independent if there is no edge joining them.

A set $S \subseteq V$ is called independent if each pair of vertices in S is independent .

Let $n(G, k) = |\{ S \subseteq V : |S|=k$ and S is independent$\}|$ be the number of independent such sets of V with k elements in G , $k\geq 0$.Then, in particular, $n(G, 0) =1$and $n(G, 1) = n$ .Without loss of generality (as far as unimodality and palindromicity is concerned) we may define the independent polynomial of G as

$$N(G) = \sum_{k\geq 0} n(G, k)\, x^k$$

It is clear that : For $p>1$

$N(K_p) = 1+px$ where K_P is complete graph with p vertices,

$N(S_4)=1+4x+3x^2+x^3$ where S_4 is star graph with 4 vertices, and

$N(K_p-e) = 1+px+x^2$ where $K_P -e$ is graph obtained by deleting an edge from K_p .

The graph G is said to be independent vertex unimodal with respect to independent polynomial if for some p , $p>0$, the inequalities **,**

$$n(G, 0) \leq n(G, 1) \leq \ldots \leq n(G, h) \geq n(G, h+1) \geq \ldots \geq n(G, p),$$

hold for some index h.

The graph G is said to be independent vertex palindromic with respect to independent polynomial if for some p , $p>0$, the equalities

$$n(G, k) = n(G, p-k)$$

hold for all $k = 0,1,2,\ldots,p$.

It is clear that S_4 is independent vertex unimodal and K_p-e is independent vertex unimodal and palindromic.

Gutman [2] established some results of independent vertex palindromic graphs. In this paper, we will describe some results of independent vertex unimodal or palindromic graphs with respect to independent polynomial.

2. Some Results:

Lemma 2.1:

Let $\overline{K_p}$ be the complement of K_p .Then for $p>0$

1- $N(\overline{K_p}) = (1+x)^p$

2- $N(\overline{K_{p+1}}) = N(\overline{K_p})+xN(\overline{K_p})$

Proof:

Part (1) we can easy prove it by using binomial theorem and by letting

$$n(G,0)=\binom{n}{0},\ n(G,k)=\binom{n}{1},\ldots,\ n(G,k)=\binom{n}{n}$$

Part (2) by part (1) $N(\overline{K_{p+1}}) = (1+x)^{p+1}$

$$= (1+x)^{p}(1+x)$$
$$= (1+x)^{p}+x(1+x)^{p}$$
$$= N(\overline{K_p})+xN(\overline{K_p}) .$$

Theorem 2.1:

For $p>1$, $\overline{K_p}$ is independent vertex unimodal and palindromic with $N(\overline{K_p}) = (1+x)^p$

Proof:

We prove this by mathematical induction on number of vertices p.

The result is true for $p=2$ since

$$N(\overline{K_2}) = 1+2x+x^2$$

We assume that the result is true for $p = k$

$$i.e.\ N(\overline{K_k}) = (1+x)^k$$

We to prove that the result is true for $p = k+1$

By lemma 2.1 $N(\overline{K_{k+1}}) = N(\overline{K_k})+x\,N(\overline{K_k})$ and by assume $N(\overline{K_k})$, $x\,N(\overline{K_k})$ are palindromic. Therefore $N(\overline{K_k})+x\,N(\overline{K_k})$ is palindromic.

Notation:

Let G_1 and G_2 be tow graphs then there is the graph $G_1 \blacktriangledown G_2$ obtained by joining each vertex of one to the vertex of the other.

Theorem 2.2:

$$N(G_1 \blacktriangledown G_2) = N(G_1) + N(G_2) - 1$$

Proof:

Since $n(G_1 \blacktriangledown G_2, 0) = n(G_1, 0) = n(G_2, 0) = 1$

Then $n(G_1 \blacktriangledown G_2, 0) = n(G_1, 0) + n(G_2, 0) - 1$

For each $k > 0$

$$n(G_1 \blacktriangledown G_2, k) = n(G_1, k) + n(G_2, k)$$

Because each independent such sets any vertex of G_1 adjacent to all vertices of G_2 consequently subsets S of $G_1 \blacktriangledown G_2$ with k elements must be $S \subseteq G_1$ or $S \subseteq G_2$ but has no pair each one in deferent graph because they are joint(dependent).Then S will be either an independent such set of G_1 or independent such set of G_2.For the number of independent such set is equal to independent such set in G_1 and independent such set in G_2. Hence

$$\begin{aligned} N(G_1 \blacktriangledown G_2) &= n(G_1 \blacktriangledown G_2, 0) + \sum_{k\geq 1} n(G_1 \blacktriangledown G_2, k)x^k \\ &= 1 + \sum_{k\geq 1} [n(G_1, k) + n(G_2, k)]x^k \\ &= 1 + \sum_{k\geq 1} n(G_1, k)x^k + \sum_{k\geq 1} n(G_2, k)x^k \\ &= \sum_{k\geq 0} n(G_1, k)x^k + \sum_{k\geq 1} n(G_2, k)x^k \\ &= \sum_{k\geq 0} n(G_1, k)\, x^k + \sum_{k\geq 0} n(G_2, k)\, x^k - 1 \end{aligned}$$

$$=N(G_1)+N(G_2)-1.$$

Corollary 2.1:

$$N(G \blacktriangledown K_p) = N(G)+N(K_p)-1$$

$$= N(G)+nx$$

Corollary 2.2:

$$N(K_p-e \blacktriangledown K_q) = N(K_p-e)+ N(K_q)-1$$

$$=1+(p+q)x+ x^2$$

Theorem 2.3:

Let G be a graph with independent polynomial

$$N(G) =1+px+ax^2$$

Then there exists independent vertex palindromic a graph $\hat{G}$ with

$$N(\hat{G}) =1+px+x^2$$

Proof :

It is clear that $1 \leq a \leq p$

Choose a pair of vertices of G and joint all other pairs we obtained the graph $\hat{G}$ with the same vertex set and having a unique independent pair so

$n(\hat{G}, 2) = 1$.Therefore $N(\hat{G}) =1+px+x^2$

Theorem 2.4:

Let G be a graph with independent polynomial

$$N(G) = 1 + px + ax^2 + x^3 \quad \text{where } a < p$$

Then there exists independent vertex palindromic graph G-e with

$$N(G\text{-}e) = 1 + px + px^2 + x^3$$

Proof :

Let $S=\{v_1, v_2, v_3\}$ be the unique independent such set of G, since
$a < p$ .Then p-a>0 deleting $e=p$-a edges from G such that $S=\{v_1, v_2, v_3\}$ the unique independent such set we obtained the graph G-e with the same vertex set and having the unique independent $S=\{v_1, v_2, v_3\}$. Therefore

$$N(G\text{-}e) = 1 + px + px^2 + x^3$$

Corollary 2.3:

Let G be a graph with independent polynomial

$$N(G) = 1 + px + ax^2 + x^3$$

Then G is a unimodal graph, and if $a<p$, then $G \blacktriangledown K_{p-a}$ is also independent vertex unimodal.

Proof :

It is clear every graph with independent polynomial

$$N(G) = 1+px+ax^2+x^3$$

is a unimodal graph. Since

$$N(K_{p-a}) = 1+(p-a)x \quad \text{then by theorem 2.1}$$

$$N(G \blacktriangledown K_{p-a}) = (1+px+ax^2+x^3) + (1+(p-a)x)-1$$

$$= 1+(2p-a)x+ax^2+x^3$$

Theorem 2.5:

Let G be a graph with unimodal independent polynomial

$$N(G) = 1+px+ax^2+x^3 \quad \text{where} \quad a > p$$

Then there exists independent vertex palindromic a graph $\hat{G}$ with

$$N(\hat{G}) = 1+ax+ax^2+x^3$$

Proof :

Since $a > p$ .Then $a-p>0$ and by theorem 2.2, and let $\hat{G} = G \blacktriangledown K_{a-p}$. Then

$$N(\hat{G}) = (1+px+ax^2+x^3) + (1+(a-p)x)-1$$

$$= 1+ax+ax^2+x^3$$

Theorem 2.2:

Let G be a graph with unimodal independent polynomial

$$N(G) = 1 + px + ax^2 + bx^3 + x^4 \quad \text{where} \quad b < p$$

Then there exists independent vertex palindromic graph G-e with

$$N(G\text{-}e) = 1 + px + cx^2 + px^3 + x^4$$

Proof :

Let $S=\{v_1, v_2, v_3, v_4\}$ be the unique independent such set of G, since $b < p$ then $p-b>0$ deleting $e= p-b$ edges from G such that
$S=\{v_1, v_2, v_3, v_4\}$ the unique independent such set and increase the number of 3- independent sets we obtained the graph G-e with the
same vertex set and having a unique independent $S=\{v_1, v_2, v_3, v_4\}$. Therefore

$$N(G\text{-}e) = 1 + px + cx^2 + px^3 + x^4$$

Theorem 2.6:

Let G be a graph with unimodal independent polynomial

$$N(G) = 1 + px + ax^2 + bx^3 + x^4 \quad \text{where} \quad b>p$$

Then there exists independent vertex palindromic a graph $\hat{G}$ with

$$N(\hat{G}) = 1 + bx + ax^2 + bx^3 + x^4$$

Proof :

Since $b > p$ then b-p>0 and by theorem 2.2 let $\hat{G}= G \blacktriangledown K_{b-p}$ then

$$N(\hat{G}) =(1+px+ax^2+bx^3+ x^4)+(1+(b-p)x)-1$$

$$=1+bx+ax^2+bx^3+x^4$$

Notation:

Construct a graph $J_{n,m,r}$ in the following meaner. Let K_1 , K_n, K_m , K_r , be the complete graphs , whose vertex set disjoint . Let n>1, m>1, r>1.Connect the (unique) vertex of K_1 ,with n-1 vertices of K_n, with m-1 vertices of K_m and with r-1 vertices of K_r .The graph $J_{n,m,r}$ thus obtained has a unique independent vertex set $S=\{v_1, v_2, v_3, v_4\}$,and by construct the graph $J_{n,m,r}$ we get

$n(J_{n,m,r} ,0) =1$, $n(J_{n,m,r} ,1) = n+m+r+1$,

$n(J_{n,m,r} ,2) = nr+rm+nm+3$, $n(J_{n,m,r} ,3) = nrm+3$,and

$$n(J_{n,m,r} ,4) =1$$

Therefore we can to verify that for n>2, m>2, r>2

$$n(J_{n,m,r} ,1) \leq n(J_{n,m,r} ,2) \leq n(J_{n,m,r} ,3) .$$

Corollary 2.4:

Let n>1, m>1, r>1.Then $\hat{G} = J_{n,m,r} \blacktriangledown K_{nrm-n-r-m+2}$ is independent vertex palindromic.

Proof :

Since $N(K_{nrm-n-r-m+2}) = 1+(nrm-m-r+2)x$, and

$$N(J_{n,m,r}) = 1+(n+m+r+1)x+(nr+rm+nm+3)x^2+(nrm+3)x^3+x^4$$

ByTheorem2.2

$$N(\hat{G}) = 1++(nrm+3)x++(nr+rm+nm+3)x^2+(nrm+3)x^3+x^4$$

Refences

[1] J.W.Kennedy; palindromic graphs,[GTN XX:8],Graph Theory Notes of New Yrork, **XXII**, The New York Academy of Sciences,27-32(1992).

[2] I. Gutman independent vertex palindromic graphs ,[GTN XXIII:3],Graph Theory Notes of New Yrork , **XXIII**, The New York Academy of Sciences , 21-24(1992).

[3] I. Gutman some properties of the Wiener polynomial ,[GTN XXV:2],Graph Theory Notes of New Yrork, **XXV**, The New York Academy of Sciences,13-18(1993).