Taiz University Research Journal. V.1, 1998, P. 12 - 24.

ON GENERALIZATION OF BOCHNER INTEGRAL

AL-HAMZA MAHMOUD

Department of Mathematics, Faculty of Science, Taiz University

ABSTRACT

Generalization of Bochner integral on the segment with conservation of many Properties of Bochner integral is considered, in particular differentiability almost everywhere of integral on Superior limit from integrable function and completeness of integrable function space are established.

INTRODUCTION

We will study a Generalization of Bochner integral on the interval with conservation of basic Properties of Bochner integral, as (i) differentiability of integral almost everywhere (a. e.) with respect to its Superior limit and (i i) completeness of integrable function space.

We note that Bochner integral is the same Lebesgue integral for vector- valued functions. We will give the concept of integral of function based on Lebesgue measurable set from real axis, when the values of function in locally convex space (L.C.S.).

The new integral satisfies two important Properties of Bochner integral: completeness of space of equivalence classes of integrable functions and differentiability of integral a. e. on superior limit. That Properties allow us to develop the theory of differential equations used in optimal control problems when the right part of differential equation is random and the initial datas is also random [1], [2], [3], [4].

There are some functions not Bochner integrable on any segment, but they are integrable under the new concept of integral.

1. The following problems, tend to a new concept of integral.

Let the Cauchy problem be given:

$$x = f(t, x) \qquad (1.1)$$

$$x(t_o) = x_o$$

(1.2) Where t, $t_o \in [\alpha, \beta] \subset R$ and f satisfies some conditions.

We assume that $x_o = x_o(\omega)$ a random quality, (i. e. it is a measurable function from probability space (Ω, Σ, P)). Then (1.1) – (1.2) takes the following form:

$$(1.1)x(t, \omega) = f(t, x(t, \omega)), \qquad (1.3)$$

$$(1.2)x(t_o, \omega) = x_o(\omega), \qquad (1.4)$$

where the solution of this problem is random process $x(t, \omega)$. In the following example given below there is not Bochner integrable function:

Let $\Omega =]0, 1[$ be a Lebesgue measurable and $f(t, x) = \sin(t x)$, where

$$f(t, .): \{x(\omega)\} \rightarrow \{\sin(t x(\omega))\}$$

is a mapping of Lp (0,1) = Lp to Lp $1 \leq p < \infty$ for fixed t , $t \neq 0$ which is non differentiable (in Frechet) at any point [5]. But for $1 < p < \infty$ we have , that $f(t, .)$ from (1.5) is differentiable on Lp on bounded subsets [6] as mapping Lp to (Lp , σ) where ($p^{-1} + p'^{-1} = 1$) $\sigma = \sigma(Lp, Lp')$ and σ

a weak topology on Lp , then for $1 \leq p < \infty$ the mapping $f(t,.)$ of Lp to Lp is differentiable on compact subsets [7]. In this case :

$$(1.6) \quad [f'_x(t, x)h](\omega) = [t\cos(tx(\omega))]h(\omega)$$

for $1 \leq p < \infty$ and

$$(1.7) \quad \|f'_x(t, x)\| = \text{ess sup}\{|t\cos(tx(\omega))| \mid \omega \in]0,1[\}$$

We note that if $\tilde{x}(.) \in Lp$ and ess sup{ $|\tilde{x}(\omega)| \mid \omega \in]0, 1[$ }$= \infty$, then

$$(1.8) \; t \longmapsto f'_x(t, \tilde{x})$$

is a mapping of R in L (Lp , Lp) which is a continuous linear mapping space from Lp to Lp with natural norm (see (1.7)) not Bochner integrable on any interval, because it does not possess the Louzin property [8] .

We remark that the optimal control problems in nfinite-dimensional spaces use the derivative concept of Frechet and Bochner integral [9] ,[10] , [11] , [12].

In order to insert the random variables in these problems, we sometimes can't use the Frechet derivative and known Bochner integral. And we need to study this problems in topological vector (linear) spaces.

2. Integral and its properties

2.1.

We denote by L (X , Y) the linear space of sequentially continuous linear mappings from topological linear space (T. L. S.) X to T. L. S. Y and X′ -linear space of continuous linear functionals on X.

P(X) – system of all continuous seminorms in X .

b (X) – system of all bounded subsets in X , and $I = [\alpha, \beta] \subset R$.

M(I) – set of all Lebesgue measurable subsets of I , $E \in M(I)$.

X – normed space, $B = \{ x \in X \mid \|x\| \leq 1 \}$

θ - Separable locally convex topology on X which satisfies the following conditions:

B_θ - sequentially complete . (2.1)

B- closed in X_θ. (2.2)

$b(X) \subset b(X_\theta)$ (2.3)

We note, that from (2.1) – (2.3) , we obtain completeness of X and that the condition (2.2) is equivalent to lower semicontinuous of norm in X as function on X_θ. From (2.3) we will have the inclusion.

(2.4) $$(X_\theta)' \subset L(X_\theta, R) \subset X'$$

The following are the examples of the spaces mentioned above:

1) X – sequentially weak complete Banach space and -$\theta = \sigma(X, X')$ weak topology on X.

2) $X = L(Y, Z)$, where Z ,Y Banach spaces -θ strong operator topology on X.

2.2. Let the function $\varphi : E \to X_\theta$ be uniformly continuous and $\varphi(E) \in b(X)$. Then exists

$$(2.5)\ \int_E \varphi(t)\,dt = \lim_{\max \Delta t_i \to 0} \sum_{i=1}^{n} \varphi(\xi_i)\, m(\,]t_{i-1}, t_i[\, \cap E)$$

Where $\alpha = t_o < t_1 < \ldots < t_n = \beta$ and $\xi_i \in\,]t_{i-1}, t_i[\, \cap E$, $\Delta t_i = t_i - t_{i-1}$

and the limit in (2.5) exists in X_θ. This fact follows from the uniform continuity of the function φ, the condition $\varphi(E) \in b(X)$ and the separability of X_θ.

Proposition 2.4.

The following assertions are true:

(2.6) $\forall x' \in L(X_\theta, R) : x' \int_E \varphi(t)\,dt = \int_E x' \varphi(t)\,dt$

(2.7) $\forall x' \in L(X_\theta, R) : |x' \int_E \varphi(t)\,dt| \le \|x'\| \int_E \|\varphi(t)\|\,dt$

(2.8) $\|\int_E \varphi(t)\,dt\| \le \int_E \|\varphi(t)\|\,dt$

(2.9) $\forall p \in P(X_\theta) : p(\int_E \varphi(t)\,dt) \le \int_E p(\varphi(t))\,dt$

(2.10) $\int_E [\,c_1 \varphi_1(t) + c_2 \varphi_2(t)\,]\,dt = c_1 \int_E \varphi_1(t)\,dt + c_2 \int_E \varphi_2(t)\,dt$

for uniformly continuous functions $\varphi_i : E \to X_\theta$, φ, which satisfies the conditions $\varphi(E) \in B(X)$, $\varphi_i(E) \in B(X)$, $c_i \in R$, $i = 1, 2$.

Proof :

The property (a) (2.6) follows from (2.5) and 2.3.

(b) (2.7) follows from (2.6) and inequality $\| x' \varphi(t) \| \leq \| x' \| . \| \varphi(t) \|$

(c) (2.8) follows from (2.7) and (2.2).

(d) (2.9) and (2.10) follows from (2.5) by using the limit in inequality and equality.

Remark 2.5.

From (2.6) we obtain the additivity of integral as a function of sets because that is correct for scalar functions.

2.6. we know that a function of sets $\varphi : E \to X_\theta$ is called measurable on E [6] if : $\forall \varepsilon > 0$ there is a compact K , $K \subset E$ where $m(E \setminus K) < \varepsilon$ and the function $\varphi|_K : K \to X_\theta$ is continuous (Louzin property) and thus $\| \varphi \| |_K : K \to R$ is lower semicontinuous and finally is measurable.

By Louzin theorem there is a compact $K_1 \subset K$ where $m(K \setminus K_1) < \varepsilon$ and the function $\| \varphi \| |_{K1} : K_1 \to R$ is continuous. Then we note that $m(E \setminus k_1) < 2\varepsilon$. Thus, the function $\| \varphi \| : E \to R$ is measurable.

2.7. Assume that

$$\tilde{\Lambda}_p(E, X_\theta) = \{ \varphi : E \to X_\theta \mid \varphi - \text{ measurable },$$

$$\int_E \| \varphi(t) \|^p dt < \infty \} \quad \text{for} \quad 1 \leq P < \infty ,$$

$\tilde{\Lambda}_*(E, X_\theta) = \{ \varphi \in \tilde{\Lambda}_1(E, X) \mid \sup_{t \in E} \| \varphi(t) \| < \infty \}$ and

$\tilde{\Lambda}_\infty(E, X) = \{ \varphi \in \tilde{\Lambda}_1(E, X_\theta) \mid \text{ess sup} \| \varphi(t) \| < \infty \}$

We define in those spaces the following seminorms :

(2.11) $\left| \| \varphi \| \right|_p = (\int_E \| \varphi(t) \|^p dt)^{1/p}$ and $1 \leq p < \infty$

(2.12) $\left| \| \varphi \| \right|_* = \sup_{t \in E} \| \varphi(t) \|$

(2.13) $\left| \| \varphi \| \right|_\infty = \text{ess sup}_{t \in E} \| \varphi(t) \|$

Where $|||\,.\,|||_*$ - norm.

It is obvious that $\tilde{\Lambda}_{P1}(E, X_\theta) \supset \tilde{\Lambda}_{P2}(E, X_\theta)$ for $1 \leq p_1 \leq p_2 \leq \infty$.

We remark that the continuous function $\varphi : E \to X_0$, has its image bounded In X, and the function is from $\tilde{\Lambda}_p(E, X_\theta)$ for $1 \leq p \leq \infty$.

For $\varphi \in \Lambda_1(E, X_\theta)$ we assume:

(2.14) $\int_E \varphi(t)\, dt = \lim_{n \to \infty} \int_{K_n} \varphi(t)\, dt$,

where K_n - compact in E, $\varphi|_{K_n} : K_n \to X_\theta$ is continuous (and thus φ is uniformly continuous) (see 2.2) $\varphi(K_n) \in b(X)$ and m $(E \setminus K_n) \to 0$.

Proposition 2.8.

The limit in (2.14) exists in X and is independent of the sequence K_n.

Remark 2.9.

From (2.14) and 2.4 follows that (2.6) – (2.10) hold for φ, $\varphi_i \in \tilde{\Lambda}_1(E, X_\theta)$.

Proposition 2.10.

Let $E_n \in M(I)$, $n \in N$, $E_m \cap E_n = \phi$. for $n \neq m$ and $E = \bigcup_{n=1} E_n$. the function $\varphi : E \to X_\theta$ satisfies

$\varphi |_{E_n} \in \tilde{\Lambda}_1(E_n , X_\theta)$ and $\sum_{n=1}^{\infty} \int_{E_n} \| \varphi(t) \| dt < \infty$. Then $\varphi \in \tilde{\Lambda}_1 (E , X_\theta)$ and

$$(2.15) \qquad \int_E \varphi(t)\,dt = \sum_{n=1} \int_{E_n} \varphi(t)\,dt$$

Where the series in (2.15)is convergent in X.

2.11. By $\Lambda_p (E , X_\theta)$, $1 \leq p < \infty$ we will denote the space of the equivalence classes from $\tilde{\Lambda}_p (E , X_\theta)$ (two functions being equivalent if they are different only on set with zero measure). We define the integral on $\Lambda_p (E , X_\theta)$ by the known way :

on each equivalence class the integral equals to integral of representative of that class and the integral is independent of that representative (This follows from 2.9) . We define in $\Lambda_p (E , X_\theta)$ the norms :

$$(2.16) \qquad ||| \varphi |||_p = \left(\int_E \| \varphi(t) \|^P dt\right)^{1/p} \text{ and } 1 \leq p < \infty$$

$$(2.17) \qquad ||| \bar{\varphi} |||_\infty = \operatorname{ess\,sup}_{t \in E} \| \bar{\varphi}(t) \|,$$

where φ is representative of class φ .

Theorem 2.12.

The space $\Lambda_1 (E , X_\theta)$ is complete under the norm (2.16) for $P = 1$.

Similarly, we can prove that the space $\Lambda_p(E, X_\theta)$ is complete, where $1 \leq p < \infty$. The completeness of space $\Lambda_*(E, X)$ follows from (2.12), where θ is considered as normed topology.

Lemma 2.13.

Let $\varphi, \varphi_n \in \tilde{\Lambda}_1(E, X_\theta)$ and $\varphi_n \to \varphi$ in $\tilde{\Lambda}_1(E, X_\theta)$. Then there is subsequence φ_{nk}, where $\varphi_{nk}(t) \to \varphi(t)$ a.e. in X.

Theorem 2.14.

Suppose $\varphi_n \in \tilde{\Lambda}_1(E, X_\theta)$, $n \in N$, $\psi \in \tilde{\Lambda}_1(E,R)$ and

$\|\varphi_n(t)\| \leq \psi(t)$ a. e. and $\varphi_n \to \varphi : E \to X_\theta$ in X a. e.

Then

(2.18) $\varphi \in \tilde{\Lambda}_1(E, X_\theta)$ and $\int_E \|\varphi_n(t) - \varphi(t)\| dt \to 0$

Theorem 2.15.

Let $\varphi, \varphi_n \in \tilde{\Lambda}_1(E, X_\theta)$, $n \in N$, $\varphi_n(t) \xrightarrow{X_\theta} \varphi(t)$ a. e. and for each $p \in P(X_\theta)$ $\exists\, \psi_p \in \tilde{\Lambda}_1(E, R)$ where $p(\varphi_n(t)) \leq \psi_p(t)$ a.e. .Then

(2.19) $\forall p \in P(X_\theta) : \int_E p(\varphi_n(t) - \varphi(t)) dt \to 0$

Corollary 2.16.

Let $\varphi_n, \varphi \in \tilde{\Lambda}_1(E, X_\theta)$, $n \in N$, $\varphi_n(t) \to \varphi(t)$ in X_θ a. e., $\psi \in \tilde{\Lambda}_1(E, R)$, $\|\varphi_n(t)\| \leq \psi(t)$ a. e. Then the formula (2.19) holds.

Theorem 2.17.

Let $\varphi \in \tilde{\Lambda}_1 (E , X_\theta)$. Then $\forall\ \varepsilon > 0$ there is a continuous function $\psi : I \to X_\theta$ where $\psi (I) \in b(X)$, $\int_E \| \varphi (t) - \psi(t)\|\, dt < \varepsilon$ and $m\{t \in E \mid \varphi (t) \neq \psi (t) \} < \varepsilon$.

If $\varphi \in \tilde{\Lambda}_\infty (E , X_\theta)$, then ψ may be selected to satisfy the condition :

$$\text{ess sup } \{\| \psi (t) \| \mid t \in I \} \leq \text{ess sup } \{\| \varphi (t) \| \mid t \in E \}$$

Corollary 2.18.

For any function $\varphi \in \tilde{\Lambda} (E , X_\theta)$ there is a sequence of continuous functions $\varphi_n : I \to X_\theta$ where $\varphi_n (I) \in b (X)$, $\varphi_n (t) \to \varphi (t)$ in X a. e. and

$$\int_E \| \varphi_n (t) - \varphi (t) \|\, dt \to 0 \text{ and } m \{ t \in E \mid \varphi_n (t) \neq \varphi (t)\} \to 0.$$

If $\varphi \in \tilde{\Lambda}_\infty (E , X_\theta)$, then φ_n may be selected where

$$\underset{t \in I}{\text{ess sup}} \| \varphi_n (t) \| \leq \underset{t \in E}{\text{ess sup}} \| \varphi (t) \|$$

Remark 2.19.

The space $\tilde{\Lambda}_1 (E , X_\theta)$ contains the closed subspace $\tilde{\Lambda}_1 (E , X)$ of Bochner integrable functions $\varphi : E \to X$. The space $\Lambda_1 (E , X_\theta)$ contains All continuous functions $\varphi : E \to X_\theta$ where $\varphi (E) \in b (X)$ (see 2.7). moreover the space contains the functions from (1.8), which do not be Bochner integrable, and thus $\tilde{\Lambda}_1 (E , X_\theta) \neq \tilde{\Lambda}_1 (E ,X)$.

We also remark that integrable functions as in 2.7. will be integrable under the Pettis concept [6].

2.20. For $\varphi \in \Lambda_1(I, X_\theta)$ and $t, t_1 \in I$, and $t < t_1$ we assume $\int_t^{t'} \varphi(s)\,ds = -\int_{t'}^{t} \varphi(s)\,ds$

Theorem 2.21.

Let $\varphi \in \tilde{\Lambda}_1(I, X_\theta)$. Then the function $\phi(t) = \int_\alpha^t \varphi(s)\,ds$ is differentiable a. e. as a map from I to X_θ and $\phi'(t)=\varphi(t)$ a. e.

2.22. We note that the function $\phi : I \to X$ is called absolutely continuous if $\forall \varepsilon > 0$, $\exists\, \delta > 0$ such that for every finite disjoint open subinteivals $]\alpha_i, \beta_i[\subset I$, the sum of whose lengths is less than δ holds the inequality :

$$\sum_1 \| \phi(\beta_i) - \phi(\alpha_i) \| < \varepsilon$$

Proposition 2.23.

Let $\varphi \in \tilde{\Lambda}_1(I, X_\theta)$. Then the function $\phi(t) = \int_\alpha^t \varphi(s)\,ds$ is absolutely continuous as mapping from I to X .

Proposition 2.24.

Let $\varphi : I \to X_\theta$ be a differentiable function a. e. , where φ as map from I to X_θ is absolutely continuous and $\varphi'(t) = 0$ a. e., Then $\varphi(t)$ is a constant function.

2.25. Assume

$$W^1_p(I, X_\theta) = \{ \phi : I \to X_\theta \mid \exists\, \varphi \in \tilde{\Lambda}_p(I, X_\theta)\ \exists a \in R : \phi(t) = a + \int_\alpha^t \varphi(s)\,ds \}.$$

For $1 \le p \le \infty$. From 2.21 follows that for every $\phi \in W^1_p(I, X_\theta)$ the function $\varphi \in \tilde{\Lambda}_p(I, X_\theta)$ will be defined in a unique way and that is sufficient for $\phi'(t)$ to give an element from $\tilde{\Lambda}_p(I, X_\theta)$ in a unique way.

Proposition 2.26.

Let $\phi \in W^1_p(I, X_\theta)$. Then $\phi(t) - \phi(t_1) = \int_{t_1}^{t} \phi'(s)\,ds$ this follows from 2.25, 2.21 and 2.11.

Proposition 2.27.

The two following conditions are equivalent :

1) $\phi \in W^1_p(I, X_\theta)$,

2) $\phi : I \to X$ is absolutely continuous and $\phi : I \to X_\theta$ is differentiable a. e., and $\exists \varphi \in \tilde{\Lambda}_p(I, X_\theta)$ such that $\phi'(t) = \varphi(t)$ a. e.,

REFERENCES

AL-Hamza, M., 1992. The problem of optimal control in infinite-dimensional spaces. The journal of sciences, Altahaddi University, Libya, No 1.

AL-Hamza, M., 1985. The pontryagin's maximum principle and transversal conditions in infinite-dimensional space "unpublished dissertation Moscow.

AL-Hamza, M., and Sukhinin, M. F., 1984. The differentiability theorem of ordinary differential equations solution at initial condition Moscow, 1984, in Zbornik of scientific works differential equations and functional analysis. P. 98-140

Sukhinin, M. .F., 1986. On infinite-dimensional differential equations with non mesurable by Bochner right part and necessary conditions of optimality. Institute of teknical and scientific informations , Moscow, No 1351-B86, 130 p .

Krasnoselski, M. A., 1966. Integral operators in integrable function spaces . Moscow, 499 p.

Averbukh, V. I, Smolyanov, O. H., 1986. Uspekhi Mat. Nauk, 23, N.4. 67p, .Moscow.

Sova, M. C., 1966. math. Journal, 16, N3, 339p.

Edwards, R.1969. Functional Analysis. Moscow, Mir., 1072p.

Yosida, K., 1965. Functional Analysis. springer ;New York.

Sukhinin, M. .F., 1978. Mat. Sbornik, 106, N3 Moscow.

Sukhinin, M. F., 1985. Different. Equations, 21, N6, Moscow.

Shilov, H., Gurevich, B., 1967. Integral, Measure and derivative. Moscow.