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(D-preconnected Sets in D-Metric Spaces)

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Abstract:
The purpose of this paper is to introduce and investigate a weaker form of D-connected set in D-metric spaces, namely D-preconnected set. The relationships among these sets with the other known sets are introduced. Furthermore, we give some properties of D-preconnceted sets.

Keywords: Closed set; Metric spaces, Connceted sets

1. Introduction

The study of ordinary metric spaces is fundamental in topology and functional analysis. Dhage, [3], introduced a new notion of a new notion and structure of D-metric space which is a natural generalization of the notion of ordinary metric space to higher dimensional metric spaces. Dhage, [2], introduced some results in D–metric spaces and gave the notions of both D-open and D-closed balls. Ali Fora, et.al, [1], introdused and a new topological structure of D-closed set, D-continuous, D-connected and D-fixed point property. They also discussed and established some results of these subjects. Some fixed point results applied in D-metric spaces in [6, 5]. In ,[7], Al-shami introduced the notions on somewhere dense sets and ST_1–spaces. The class of somewhere dense sets contains all preopen, regular open, semi open, α-open, β-open, b-open and β-open sets with the exception of the empty set. Hussain and Saif, [8], introduced and investigated weak form of D-open sets in D-metric spaces, namely D-preopen sets. The relationships among this form with the other known sets are introduced.

They introduced the notions of the interior operator, the closure operator and frontier operator via D-preopen sets. Hussain and Saif, [9], introduced the concept of D-precontinuous function by utilizing the D-preopen sets. Hussain and Saif, [10], introduced the concept of contra and almost D-precontinuous functions via D-preopen sets. Hussain and Saif, [11], introduced and emphasized strong form of D-compact sets in D-metric spaces, namely D-precompact sets and they introduced the notions of sequentially D-precompact sets.

In this work, we investigate and introduce a new class of D-connected in D-metric spaces by using D-preopen sets, say the class of D-preconnected Also we give some properties of this class.

2. Preliminaries

Definition 2.1. [4]. A nonempty set X, together with a function $D : X \times X \times X \to [0,\infty)$ is called a D-metric space, denoted by (X,D) if D satisfies the following conditions for each $x,y,z,a \in X$:

1. $D(x,y,z) = 0 \to x = y = z$ (coincidence);
2. $D(x,y,z) = D(p(x,y,z))$, where p is a permutation of x,y,z (symmetry)
3. $D(x,y,z) \leq D(x,y,a) + D(x,a,z) + D(a,y,z)$ for all $x,y,z,a \in X$ (tetrahedral inequality).

$O_\varepsilon^D(x)$ denotes the D-open ball with center x and radius $\varepsilon > 0$, that is,

$$O_\varepsilon^D(x) = \{y \in X : D(x,y,y) < \varepsilon\}. \quad C_\varepsilon^D(x)$$

denotes the D-closed ball with center x and radius $\varepsilon > 0$, that is,

$$C_\varepsilon^D(x) = \{y \in X : D(x,y,y) \leq \varepsilon\}.$$

The set $G \subseteq X$ is called an D-open set in D-metric space (X,D) if for every $x \in G$, there is $\varepsilon > 0$ such that $O_\varepsilon^D(x) \subseteq G$. The set G is called D-closed set in D-metric space (X,D) if $X - G$ is D-open set in D-metric space (X,D).

Theorem 2.2. [1]. Every finite set in a D-metric space (X,D) must be a D-closed set.

Definition 2.3. [2]. A D-metrizable topological space (X,τ_D) is a topological space on X which has basis $\{O_\varepsilon^D(x) : x \in X, \varepsilon > 0\}$.

Theorem 2.4. [2]. If a topological space X is metrizable then it is D-metrizable.

Definition 2.5. [1]. A D-metric space (X,D) is called a D-disconnected space provided there exists a partition $\{A,B\}$ for X consisting of tow non-empty D-closed sets in X, otherwice it is called a D-connected space.

Remark 2.6. [1]. A subset E of a D-metric space (X,D) be a D-closed if $E = cl_D(E)$ (equivalently $cl_D(E) \subseteq E$), i.e. if for any $\{x_n\} \in E$ and $p \in X$, if $\{x_n\} \to -Dp$, then $p \in E$. The complement of a D-closed set is called Dopen set in (X,D).

Remark 2.7. [1]. An intersection of any collection of Dclosed sets in a D-metric space (X,D) is a D-closed set.

Definition 2.8. [1]. Let (X,D) be a D-metric space and A, B be two nonempty subsets of X. Then A and B are called D-separated sets provided $A \cap Cl_D(B) = B \cap Cl_D(A) = \emptyset$.

Lemma 2.9. [1]. In a D-metric space (X,D), if A and B are two D-separated sets in X such that $A \cup B = X$, then both of them is a D-clopen set in X.

Definition 2.10. [1]. Let $f : (X,D) \longrightarrow (Y,P)$ be a function between two D-metric spaces. Then f is said to be D_P-continuous at $p \in X$ provided that for any sequence $\{x_n\}$ converge in (X,D) to p, then $\{f(x_n)\}$ must converges in (Y,P) to $f(p)$. A function $f : (X,D) \longrightarrow (Y,P)$ is called D_P-continuous if f is D_P-continuous at each p in X. In the case $X = Y$ and $D = P$, we write
D-continuous instead of D_P-continuous function.

Theorem 2.11. [1]. The following statements are equivalent for a D-metric space (X,D).

1. (X,D) is a D-disconnected space.
2. There exists a D-clopen set A in X such that $\emptyset$ 6= A 6= X.
3. X has a partition consisting of two nonempty Dopen sets.
4. X has a nontrivial D-separation.
5. There exists a surjective D_p-continuous function $f : (X,D) \to (\{0,1\},p)$, where p is a D-metric on $\{0,1\}$.

For a D-metric space (X,D), the set $G \subseteq X$ is called a D-preopen set, [8], in D-metric space (X,D) if for every $x \in G$, there is $\delta > 0$ such that for every $y \in O_\delta^D(x)$, $O_\varepsilon^D(y) \cap G \neq \emptyset$ for every $\varepsilon > 0$. The set $G \subseteq X$ is called a D-preclosed set in a D-metric space (X,D) if $X - G$ is a D-preopen set in D-metric space (X,D). Recall, [8], every D-open set is a D-preopen set. For a D-metric space (X,D)

and $G \subseteq X$, $Cl_P{}^D(G)$ is a D-preclosed set. For a D-metric space (X,D) and $G \subseteq X$, we have $G \subseteq Cl_P{}^D(G)$.

Theorem 2.12. [8]. For a D-metric space (X,D) and $G \subseteq X$, $Cl_P{}^D(G) = G$ if and only if G is a D-preclosed set.

Definition 2.13. [8]. Let (X,D) be a D-metric space and $G \subseteq X$. The D_P-closure operator of G is denoted by $Cl_P{}^D(G)$ and defined by

$Cl_P{}^D(G) = \cap\{H \subseteq X : G \subseteq H$ and H is D-preclosed set$\}$.

Definition 2.14. [9]. Let $f : (X,D) -\!\!\rightarrow (Y,D^0)$ be a function between two D-metric spaces (X,D) and (Y,D^0). Then f is called a D-precontinuous if $f^{-1}(U)$ is a D-preopen set in (X,D) for every D^0-open set U in Y.

Definition 2.15. [10]. Let $f : (X,D) -\!\!\rightarrow (Y,D^0)$ be a function between two D-metric spaces (X,D) and (Y,D^0). Then f is called a contra D-precontinuous function if $f^{-1}(V)$ is a D-preclosed set in (X,D) for every D^0-open set V in Y**3.**

3. D-preconnected sets

Definition 3.1. Let (X,D) be a D-metric space and A,B be two nonempty subsets of X. The sets A and B are called a D_p-separated sets if $Cl_P^D(A) \cap B = \emptyset$ and $A \cap$ $Cl_P{}^D(B) = \emptyset$.

Remark 3.2. Let (X,D) be a D-metric space. Then 1. Any D_p−separated sets are disjoint sets, because

$A \cap B \subseteq A \cap Cl_P^D(B) = \emptyset$.

2. Any two nonempty D-preclosed sets in X are D_p−separated if they are disjoint sets.

Definition 3.3. A D-metric space (X,D) is called a D_p−disconnected space if it is the union of two D_p−separated sets a comma otherwise (X,D) is called a D_p−connected space.

Example 3.4. Any finite D-metric space (X,D) is a $D_p$disconnected space.

Theorem 3.5. Any D-metric space (X,D) with a finite set X is a D_p-disconnected space if X has more then one point.

Proof. A space the proof of the theorem is clear because every set in X is D-preclopen set.

Theorem 3.6. Every D-disconnected space is a $D_p$disconnected space.

Proof. The proof of the theorem is clear because $Cl_P{}^D(A) \subset Cl_D(A)$.

The converse of the above theorem need not be true.

Example 3.7. Let $(\mathbb{R},D)$ be a D-metric space given by

$$D(x,y,z) = max\{d(x,y),d(y,z),d(z,x)\},$$

where $(\mathbb{R},d)$ is the usual metric space on the set of real number R. R is the union of two D_p-separated sets Q and IR, where Q is the set of rational numbers and IR is the set of irrational numbers then R is a D_p-disconnected space which is not D-disconnected.

Theorem 3.8. A D-metric space (X,D) is a $D_p$disconnected space if and only if it is the union of two nonempty disjoint D-preopen sets.

Proof. Suppose that (X,D) is a D_p-disconnected space. Then X is the union of two D_p-separated sets, that is, there are two nonempty subsets A and B of X such that

$$Cl_P^D(A) \cap B = \emptyset,\ A \cap Cl_P^D(B) = \emptyset \text{ and } A \cup B = X.$$

Take

$$G = X - Cl_P{}^D(A) \text{ and } H = X - Cl_P{}^D(B).$$

We have G and H are D-preopen sets. Since B 6= $\emptyset$ and $Cl_P{}^D(A) \cap B = \emptyset$, we get $B \subseteq X - Cl_P{}^D(A)$, that is,

$$G = X - Cl_P^D(A) \neq \emptyset.$$

Similarly H 6= $\emptyset$. Since

$$Cl_P{}^D(A) \cap B = \emptyset,\ A \cap Cl_P{}^D(B) = \emptyset \text{ and } A \cup B = X,$$

$X-(G\cap H)=(X-G)\cup(X-H)=[Cl_P^D(A)]\cup[Cl_P^D(B)]=X.$ we have

That is, $G \cap H = \emptyset$.

Conversely, suppose that (X,D) is the union of two disjoint nonempty D-preopen subsets, say G and H. Take

$$A = X - G \text{ and } B = X - H.$$

We have A and B are D-preclosed sets, that is, $Cl_P{}^D(A) = A$ and $Cl_P{}^D(B) = B$. Since H 6= $\emptyset$ and $H \cap G = \emptyset$, then $H \subseteq X - G = A$, that is, A 6= $\emptyset$. Similarly B 6= $\emptyset$. Since $G \cap H = \emptyset$ and $G \cup H = X$, then

$$Cl_P{}^D(A) \cap B = A \cap B = (X - G) \cap (X - H) = X - (G \cup H)$$

$$= X - X = \emptyset.$$

Similarly , $A\cap Cl_P^D(B)=\emptyset$. Note that

$A\cup B = (X-G)\cup(X-H) = X-(G\cap H) = X-\emptyset = X.$

That is, (X,D) is a D$_p$-disconnected space.

Corollary 3.9. A D-metric space (X,D) is a D$_p$disconnected space if and only if it is the union of two non-empty disjoint D-preclosed subsets.

Proof. Suppose that (X,D) is a D$_p$-disconnected space. Then by Theorem (3.8), (X,D) is the union of two nonempty disjoint D-preopen subsets, say G and H. We have $X - G$ and $X - H$ are D-preclosed subsets. Since G 6= $\emptyset$,
H 6= $\emptyset$ and $X = G\cup H$ We have $X-G$ 6= $\emptyset$, $X-H$ 6= $\emptyset$
and

$$(X - G) \cap (X - H) = X - (G \cup H) = X - X = \emptyset.$$

Since $G \cap H = \emptyset$ we get

$$(X - G) \cup (X - H) = X - (G \cap H) = X - \emptyset = X.$$

Hence X is the union of two non-empty disjoint Dpreclosed subsets.

Conversely, suppose that (X,D) is the union of two nonempty disjoint D-preclosed subsets, say G and H. Take

$$A = X - G \text{ and } B = X - H.$$

We have A and B are D-preopen sets. Since H 6= $\emptyset$ and $H \cap G = \emptyset$, we have $H \subseteq X - G = A$, that is, A =6 $\emptyset$. Similarly B 6= $\emptyset$. Since $G\cap H = \emptyset$ and $G\cup H = X$, then

$$\begin{aligned} Cl_P^D(A)\cap B &= A \cap B = (X - G) \cap (X - H) \\ &= X - (G \cup H) \\ &= X - X = \emptyset. \end{aligned}$$

Similarly, $A \cap Cl_P{}^D(B) = \emptyset$. Note that

$A\cup B = (X-G)\cup(X-H) = X-(G\cap H) = X-\emptyset = X.$

Then by Theorem (3.8), (X,D) is a D$_p$-disconnected space

4.Some properties of D-preconnected sets

Theorem 4.1. Let (X,D) be a D-metric space the following statements are equivalent:

1. X is a D$_p$-connected space
2. The only D-preclopen (D-preclosed and D-preopen) subsets of X are $\emptyset$ and X
3. X has no D$_p$-separations
4. Every D-precontinuous map $X \to \{0,1\}$, form a space X on to to a discrete space $\{0,1\}$, is constant.

Proof. $1 \Rightarrow 2$: If C is a D-preclosed and D-preopen subset of X then also the complement $X-C$ is a D-preclosed and D-preopen sets. Thus $X = C \cup (X - C)$ is the union of two disjoint D-preopen subsets and therefore one of these sets must be empty.

$2 \Rightarrow 3$: Suppose that $X = A \cup B$ where A and B are D$_p$-separated sets. Then $Cl_P{}^D(A) \subset A$ for $Cl_P{}^D(A)$ does not meet B. Thus A is a D-preclosed set. Similarly, B is a D-preclosed set. Therefore A is both D-preclosed and D-preopen set. By hypothesis, A is either the empty set. $3 \Rightarrow 4$: Suppose that there exists a surjective Dprecontinuous map $f : X \to \{0,1\}$. Then $X = f^{-1}(0)\cup f^{-1}(1)$

is a D_p-separation of X. 4 ⇒ 1: We prove the contrapositive. Suppose that X is D_p-disconnected. Then $X = U_1 \cup U_2$ where U_1 and U_1 are non-empty disjoint, D-preopen sets. The map $f : X \to \{0,1\}$ given by $f(U_1) = 0$ and $f(U_2) = 1$ is a non-constant D-precontinuous map.

Remark 4.2. Let (X,D) be a D-metric space and $A \subset Y \subset X$. Then $Cl^D_{P_Y}(A) = Y \cap Cl^D_P(A)$.

Lemma 4.3. Let (X,D) be a D-metric space and $Y \subset X$ be a subspace. Then
Y is a D_p-connected space if and any if Y is not the union of two D_p-separated non-empty subsets of X.

Proof. Suppose that $Y = Y_1 \cup Y_2$ is the union of two subspaces. Observe that Y_1 and Y_2 are D_p-separated in Y if and any if Y_1 and Y_2 are D_p-separated in X because $Cl^D_{P_Y}(Y_1) \cap Y_2 = Cl^D_P(Y_1) \cap Y \cap Y_2 = Cl^D_P(Y_1) \cap$ Y_2. The proof of the lemma now follows immediately from theorem 4.1,the equivalence of (1) and (3).

Corollary 4.4. Let (X,D) be a D-metric space and $Y \subset X$ be a D_p-connected subspace. For every pair A and B of D_p-separated subsets of X such that $Y \subset A \cup B$ we have either $Y \subset A$ or $Y \subset B$.

Proof. The subsets $Y \cap A$ and $Y \cap B$ are D_p-separated (because the bigger sets A and B are D_p-separated) with union Y . By lemma(4.3), one of them must be empty, $Y \cap A = \emptyset$, say, so that $Y \subset B$

Theorem 4.5. Let (X,D) be a D-metric space $Y \subset X$ be a subspace and $(Y_j) : j \in J$ be an indexed family of D_p-connected subspaces of the space X. Suppose that there is a $j_0 \in J$ such that Y_j and Y_{j0} are not D_p-separated for any $j \in J$. Then the union $\cup_{j\in J} Y_j$ is a D_p-connected set.

Proof. We have to prove that the subspace $Y = \cup_{j\in J} Y_j$ is D_p-connected. We use the criterion of Lemma(4.3). Suppose that $Y = A \cup B$ is the union of two D_p-separated subsets A and B of X. Each of the sets Y_j is by Corollary 4.4, each of the set Y_j is contained in either A or B. Let's say that Y_{j0} is contained in A. Then Y_j must be contained in A for all $j \in J$ for if $Y_j \subseteq B$ for some $j \in J$ then Y_j and Y_{j0} would be D_p-separated. Therefore $Y = A$.

Corollary 4.6. The union of a collection of D_p-connected subspaces with a point in common is D_p-connected set.

Proof. . Apply theorem 4.5 on any of the subspaces as Y_{j0}.

Corollary 4.7. Let (X,D) be a D-metric space $C \subset X$ be a subspace and C be a D_p-connected subspace of X. Then $Cl_P{}^D(C)$ is D_p-connected. Indeed, if $C \subset Y \subset Cl_P{}^D C$ then Y is D_p-connected.

Proof. . . Apply Theorem(4.5) to the collection consisting of C and $\{y\}$ for $y \in Y - C$. Then C and $\{y\}$ are not D_p-separated becauce $y \in Cl_P{}^D C$

Corollary 4.8. Let (X,D) be a D-metric space. Suppose that for any two points in X there is a D_p-connected subspace containing both of them. Then X is D_p-connected.

Proof. Let x_0 be some fixed point of X. For each point $x \in X$, let C_x be a D_p-connected subspace containing x_0 and x. Then $X = \cup C_x$ is a D_p-connected becauce
$\cap C_x$ 6= ∅ corollary(4.6).

Theorem 4.9. Let $f : (X,D) \to (Y,D^0)$ be a Dprecontinuous surjection function. If X is a D_p-connected space then Y is D-connected space.

Proof. Suppose that Y is a D-disconnected space. We have Y is the union of two non-empty disjoint D-open subsets, say A and B. Since f is a D-precontinuous we get $f^{-1}(A)$ and $f^{-1}(B)$ are D-preopen sets. Since A 6= ∅, B 6= ∅ and f is a surjection we get $f^{-1}(B)$ 6= ∅ and $f^{-1}(A)$ 6= ∅. Since $A \cap B = \emptyset$ and $A \cup B = Y$ then $f^{-1}(A) \cap f^{-1}(B) = f^{-1}(A \cap B) = f^{-1}(\emptyset) = \emptyset$

and

$$f^{-1}(A) \cup f^{-1}(B) = f^{-1}(A \cup B) = f^{-1}(Y) = X.$$

Hence X is the union of two disjoint nonempty D-preopen subsets, that is, X is a D_p-disconnected space. This is a contradiction. Hence Y is a D-connected space.

Theorem 4.10. Let f : (X,D) → (Y,D^0) be a contra Dprecontinuous surjection function. If X is a D_p-connected space then Y is not a discrete space.

Proof. Suppose that Y is a discrete space. Then there is a proper nonempty D-open and D-closed sets in Y , say A. Since f is a contra D-precontinuous and surjection we have $f^{-1}(A)$ is a proper nonempty D-preopen and Dpreclosed set in X. Which is a contradiction as X is a D_p-connected space.

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